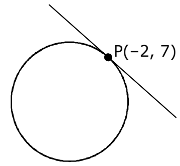
EXERCISE – IV HINTS & SOLUTIONS

Sol.1 $x + y = 5$

& point circle

$(x + 2)^2 + (y + 7)^2 = 0$



circle is

$\{x^2 + y^2 + 4x - 14y + 53\} + \lambda (x + y - 5) = 0$

$\Rightarrow x^2+y^2+x\,(4x+\lambda)+y(\lambda-14) + 53-5\lambda=0$

is orthogonal to

$x^2 + y^2 + 4x - 6y + 9 = 0$

$(4 + \lambda)\, 2 + (\lambda - 14)\,(-3) = 53 - 5\lambda$

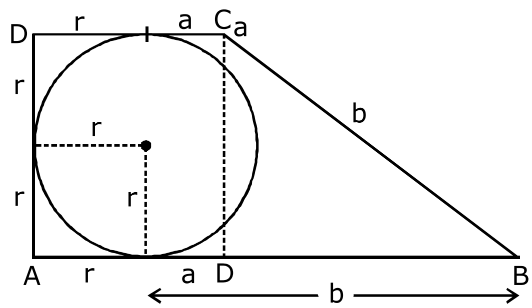
$\Rightarrow 8 + 2\lambda + 42 - 3\lambda = 62 - 5\lambda$

$\Rightarrow 4\lambda = 12 \Rightarrow \lambda = 3$

circle is $x^2 + y^2 + 7x - 11y + 38 = 0$

Sol.2 We wish to prove

$$AD = \frac{2(AB)(CD)}{(AB)+(CD)}$$



$AB = r + b$

$CD = (r + a)$

where $AD = 2R$

$$R.H.S. = \frac{2(r+a)(r+b)}{(r+a)+(r+b)}$$

$$= \frac{2[r^2+r(a+b)+ab]}{2r+(a+b)}$$

In ΔBCD . $(a + b)^2 = (2r)^2 + (b - a)^2$

$\Rightarrow a^2 + b^2 + 2ab = 4r^2 + a^2 + b^2 - 2ab$

$\Rightarrow \quad 4ab = r^2 \quad \Rightarrow \quad ab = r^2$

$$R.H.S. = \frac{2r^2+2r(a+b)+2r^2}{2r+(a+b)}$$

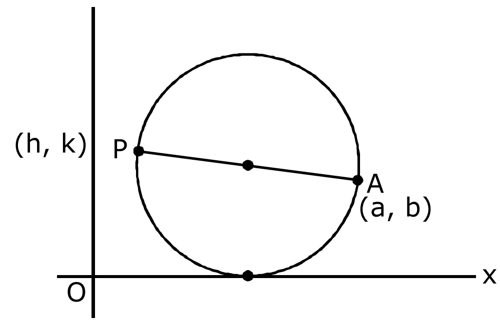
$$= \frac{4r^2+2r(a+b)}{2r+(a+b)} = \frac{2r(2r+(a+b))}{(2r+(a+b))}$$

$= 2r =$ L.H.S. $= AD$

Hence proved

Sol.3 Centre $\left(\frac{h+a}{2}, \frac{k+b}{2}\right)$

$$r = \frac{k+b}{2}$$



Point of contact is $\left(\frac{h+a}{2}, 0\right)$

touches x-axis $\Rightarrow \quad 2\sqrt{g^2 - C} = 0$

$(x - a)\,(x - h) + (y - b)\,(y - k) = 0$

$x^2 + y^2 - (a + h)\,x - (k + b) + (ab + bk) = 0$

$$g = -\frac{(h+a)}{2},\ c = (ah + bk)$$

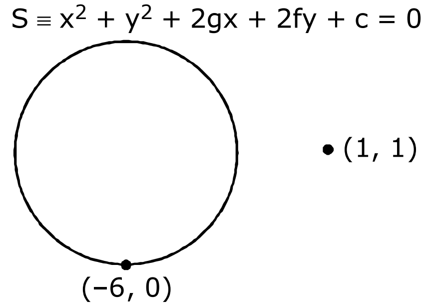
$$\Rightarrow \frac{(h+a)^2}{4} = ah + bk$$

$\Rightarrow (h - a)^2 = 4bk$

$\Rightarrow (k - a)^2 = 4by$

Sol.4 Let circle is

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$



Passing $(-6, 0)$

$36 - 12g + c = 0$ ....(i)

& $\quad S_{(1,\ 1)} = S$

$1 + 1\ 2g + 2f + c = 5$

orthogonal with $x^2 + y^2 - 4x - 6y - 3 = 0$

$-4g - 6y = C - 3$

$4g + 6f + C - 3 = 0$ ....(iii)

From (ii) & (iii)

$2g + 4f = 0 \Rightarrow g = -2f$

put in (ii)

$2g - g + C - 3 = 0$

$g + c = 3$

from (i)

$36 - 13g = -3 \Rightarrow g = 3,$

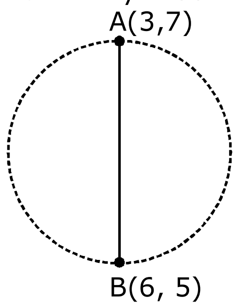
$\Rightarrow f = -\frac{3}{2}, c = 0$

$x^2 + y^2 + 6x - 3y = 0$

Sol.5 $S + \lambda L = 0$

$S \equiv (x - 3)(x - 6)(y - 7)(y - 5) = 0$

$x^2 + y^2 - 9x - 12y + 53 = 0$



$L \equiv y - 5 = -\frac{2}{3}(x - 6)$

$2x + 3y - 27 = 0$

$(x^2 + y^2 - 9x - 12y + 53)$
$\quad + \lambda(2x + 3y - 27) = 0$

$x^2 + y^2 + x(2\lambda - 9) + y(3\lambda - 12)$
$\quad + (53 - 27\lambda) = 0$

$x^2 + y^2 \;\; 4x - 6y - 3 = 0$

$(2\lambda - 5)x + (3\lambda - 6)y + (56 - 27\lambda) = 0$

$(-5x - 6y + 56) + \lambda(2x + 3y - 27) = 0$

$\Rightarrow \quad L_1 + \lambda L_2 = 0$

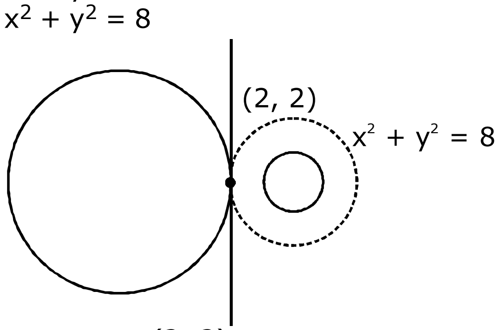
Always passes through intersection of L_1 & L_2

$x = 2$ & $y = \frac{23}{3} \Rightarrow \left(2, \frac{23}{3}\right)$

Sol.6 Director circle of

$x^2 + y^2 = 4$

$x^2 + y^2 = 8$



tangent at (2, 2)

$2x = 2y = 8 \quad \Rightarrow \quad x + y = 4$

is radical axis

$\therefore \quad (x^2 + y^2 - 8) + \lambda(x + y - 4) = 0$

passes through (1, 1)

$-6 + \lambda(-2) \quad \Rightarrow \quad \lambda = -3$

Circle is $x^2 + y^2 - 3x - 3y + 4 = 0$

Sol.7 $x^2 + y^2 + 2x - 2y = 0$

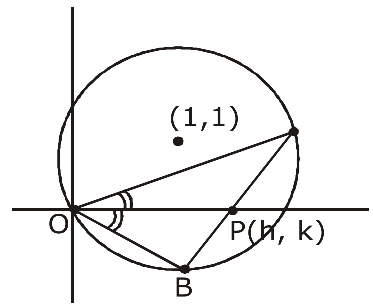
Let mid point of chord is

P(h, k)

$T = S_1$

$hx + ky - (x + h) - (y + k) = h^2 + k^2 - 2h - 2k$

$x(h - 1) + y(k - 1) = h^2 + k^2 - h - k$



homoginize with circle

$$x^2 + y^2 - 2(x + y)\left[\frac{x(h-1) + y(k-1)}{h^2 + k^2 - h - k}\right] = 0$$

angle $\angle AOP = \angle BOP$

$\Rightarrow \quad m_{OA} + m_{OB} = 0$ {coeff. of xy = 0}

$\Rightarrow \quad H = 0$

$$-\frac{[2(k-1) + 2(h-1)]}{2(h^2 + k^2 - h - k)} = 0$$

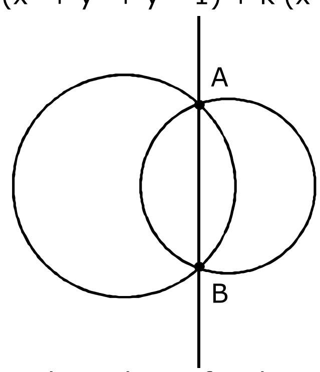
$\Rightarrow \quad k - 1 + h - 1 = 0$

$\Rightarrow \quad h + k = 2$

Locus $x + y = 2$

Sol.8 $x^2 + y^2 + kx + (1 + k)y - (k + 1) = 0$

$\Rightarrow \quad (x^2 + y^2 + y - 1) + k(x + y - 1) = 0$



passes through two fixed point for $\forall\, k \in R$

$x^2 + y^2 + y - 1 = 0$ & $\quad x + y = 1$

$x = 1 - y$

$\Rightarrow \quad (y - 1)^2 + y^2 + y - 1 = 0$

$\Rightarrow \quad 2y^2 - y = 0$

$\Rightarrow \quad y(2y - 1) = 0 \Rightarrow \; y = 0$ or $y = \frac{1}{2}$

$\therefore \quad x = 1$ or $x = \frac{1}{2}$

(i) $\quad A(1, 0)$ or $B\left(\frac{1}{2}, \frac{1}{2}\right)$

(ii) minimum radius (which circle whose diamter AB)

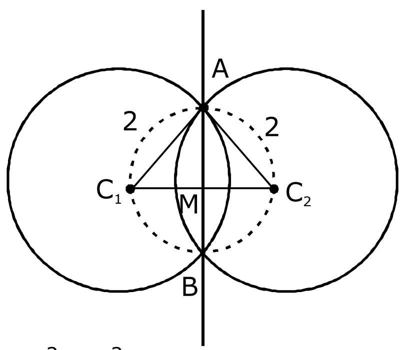
$$\frac{AB}{2} = \frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{2\sqrt{2}}$$

Sol.9 $2x^2 + 2y^2 - 2x + 6y - 3 = 0$

$$\Rightarrow \quad x^2 + y^2 - x + 3y - \frac{3}{2} = 0$$

$$\Rightarrow \quad C_1\left(\frac{1}{2}, \frac{-3}{2}\right), r_2 = 2$$



& $x^2 + y^2 + 4x + 2y + 1 = 0$

$\Rightarrow \quad C_2(-2, -1), r_2 = 2$

$$C_1C_2 = \sqrt{\frac{25}{4}+\frac{1}{4}} = \sqrt{\frac{13}{2}}$$

$$\frac{C_1C_2}{2} = C_1M = \frac{\sqrt{13}}{2\sqrt{2}} \quad (\because \quad r_1 = r_2)$$

$$M \equiv \left(\frac{\frac{1}{2}-2}{2}, \frac{-\frac{3}{2}-1}{2}\right) \equiv \left(\frac{-3}{4}, \frac{-5}{4}\right)$$

$$AM = \sqrt{2^2 - \frac{13}{8}} = \sqrt{\frac{19}{8}}$$

$$\text{Centre } \left(\frac{-3}{4}, \frac{-5}{4}\right) \text{ radius} = \left(\sqrt{\frac{19}{8}}\right)$$

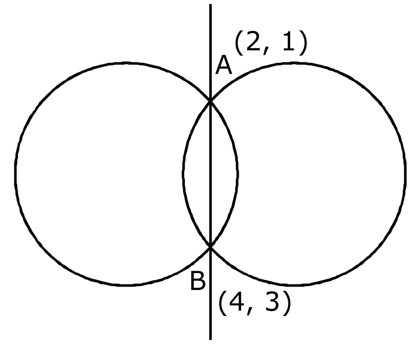
$$\left(x+\frac{3}{4}\right)^2 + \left(y+\frac{5}{4}\right)^2 = \frac{19}{8}$$

$16x^2 + 16y^2 + 24x + 4y + 9 + 25 = 38$

$\Rightarrow 16x^2 + 16y^2 + 24x + 4y + 9 + 25 = 38$

$\Rightarrow 4x^2 + 4y^2 + 6x + 10y - 1 = 0$

Sol.10 Family of circles



$S + \lambda L = 0$

$S \equiv (x - 2)(x - 4) + (y - 1)(y - 3) = 0$

$L \equiv (y - 1) = 1 (x - 2)$

$S \equiv x^2 + y^2 - 6x - 4y + 11 = 0$

& $L \equiv x - y - 1 = 0$

$(x^2 + y^2 - 6x - 4y + 11) + \lambda (x - y - 1) = 0$

$\forall \lambda \in R$

$\Rightarrow x^2+y^2+x(\lambda-6)+y(-\lambda-4) + (11 - \lambda) = 0$

Let other circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ ....(i)

$(\lambda - 6) g + (-\lambda + 4) f = 11 - \lambda + c$

$(6y + 4f + c + 11) + \lambda (-g + f - 1) = 0 \; \{\because \; \forall \lambda \in R$

$\Rightarrow 6g + 4f + c + 11 = 0$ & $f = g + 1$

$6g + 4g + 4 + c + 11$

$\Rightarrow 10g + c + 15 = 0 \quad \Rightarrow \quad c = -\log - 15$

put in equation (1)

$x^2 + y^2 + 2gx + 2(g + 1)y - 10g - 15 = 0$

$\Rightarrow (x^2 + y^2 + 2y - 15) + g(2x + 2y - 10) = 0$

$x^2 + y^2 + 2y - 15 = 0$ & $x + y = 5$

$\Rightarrow \quad y^2 - 4y + 5 = 0 \begin{matrix} y_1 \\ y_2 \end{matrix}$

& $\quad 2x^2 - 12x + 2a = 0$

$x^2 - 6x + 10 = 0 \begin{matrix} x_1 \\ x_2 \end{matrix}$

$x_1^3 + x_2^3 + y_1^3 + y_1^3 = (x_1 + x_2)^3$

$- 3x_1x_2(x_1 + x_2) + (y_1 + y_2)^3$

$- 3y_1y_2(y_1 + y_2)$

$= (4)^3 - 3.5(4) + 6^3 - 3.10(6)$

$= 64 - 60 + 216 - 180 = 220 - 180 = 40$

Sol.11 $y = x + 10$

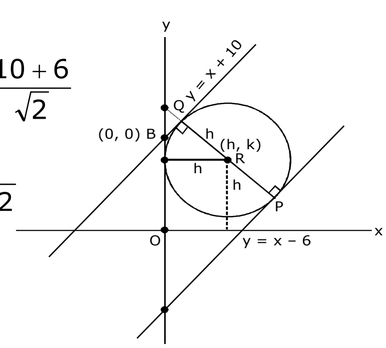
$y = x - 6$

$$2r = 2h = \frac{10+6}{\sqrt{2}}$$

$$= \frac{16}{\sqrt{2}} = 8\sqrt{2}$$

$2h = 8\sqrt{2}$

$h = 4\sqrt{2}$



$\perp$ distance equal to $h = 4\sqrt{2}$ from $(4\sqrt{2}, k)$

$$4\sqrt{2} = \frac{|4\sqrt{2} - k + 10|}{\sqrt{1^2 + 1^2}} \Rightarrow 8 = |4\sqrt{2} - k + 10|$$

{geometrically k < 10}

$8 = 4\sqrt{2} - k + 10$

$k = 10 - 8 + 4\sqrt{2}$

$k = 2 + 4\sqrt{2}$

$h + k = 2 + 8\sqrt{2}$

$h + k = 2 + 8\sqrt{2}$

$= a + b\sqrt{2}$ $\qquad a = 2, b = 8$

$\therefore \quad a + b = 10$

Sol.12 area ABCD = $900\sqrt{2}$ sq. units

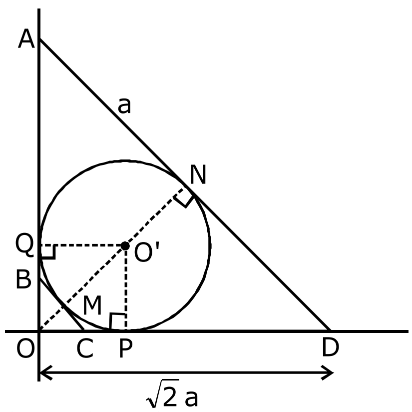
ON = ND = NA = a (let)

area $\triangle OAD = a^2$

$OD = OA = \sqrt{2}\,a$

$OP = \sqrt{2}\,a - a$

$= a(\sqrt{2} - 1)$ = radius



$OM = ON - 2r$

$= a - 2a(\sqrt{2} - 1) = a(3 - 2\sqrt{2})$

area $\triangle OBC = (OH)^2 = a^2(3 - 2\sqrt{2})^2$

$a^2 - a^2(3 - 2\sqrt{2}) = 900\sqrt{2}$

$\Rightarrow a^2[1 - (3 - 2\sqrt{2})^2] = 900\sqrt{2}$

$$\Rightarrow a^2 = \frac{900\sqrt{2}}{(1 + 3 - 2\sqrt{2})(1 - 3 + 2\sqrt{2})}$$

$$\Rightarrow \quad = \frac{900\sqrt{2}}{2\sqrt{2}(\sqrt{2} - 1)2(\sqrt{2} - 1)}$$

$$\Rightarrow a^2 = \frac{225}{(\sqrt{2} - 1)^2} \qquad \Rightarrow a = \frac{15}{(\sqrt{2} - 1)}$$

$\Rightarrow a(\sqrt{2} - 1) = 15 = r$

Sol.13 $A, B, C \in R$

(i) $(\sin A, \cos B)$ lines on $x^2 + y^2 = 1$

$\Rightarrow \sin^2 A + \cos^2 B = 1 \Rightarrow A = B$

(ii) tan C & cot C are defined.

$\Rightarrow C \neq 0$ & $C \neq \frac{\pi}{2}$

$y = (\tan C - \sin A)^2 + (\cot C - \cos B)^2$

$= \tan^2 C + \cot^2 C + \sin^2 A + \cos^2 B$

$- 2[\tan C \sin A + \cot C \cos B]$

$$= \left[\tan^2 C + \frac{1}{\tan^2 C}\right]$$

$+ 1 - 2[\tan C \sin A + \cot C \cos A]$

For min. value of y $\tan^2 C + \frac{1}{\tan^2 C}$ is 2.

$$\tan^2 C + \frac{1}{\tan^2 C} = 2$$

$$\left(\tan c - \frac{1}{\tan c}\right)^2 = 0$$

$\Rightarrow \tan C = 1 \quad \cot C = 1$

$y = 2 + 1 - 2[\sin A + \cos B]_{min}$

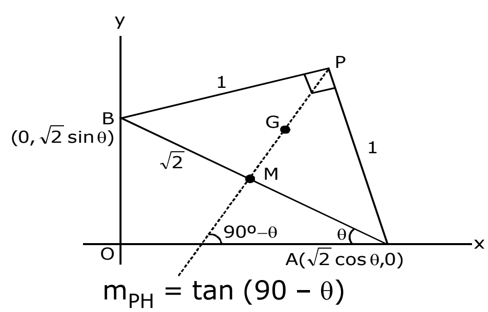
$= 3 - 2\sqrt{2} = 3 + (-2)\sqrt{2}$

$= a + b\sqrt{2} \qquad \Rightarrow a = 3, b = -2$

$a^3 + b^3 = 3^3 + (-2)^3 = 27 - 8 = 19$

Sol.14 Let G(h, k)

$$M \equiv \left(\frac{\cos\theta}{\sqrt{2}}, \frac{\sin\theta}{\sqrt{2}}\right)$$



$m_{PH} = \tan(90 - \theta)$

Line PM

$$\frac{x - \frac{\cos\theta}{\sqrt{2}}}{\sin(\theta)} = \frac{y - \frac{\sin\theta}{\sqrt{2}}}{\cos(\theta)} = +\frac{1}{3\sqrt{2}}$$

$$PM = \frac{1}{\sqrt{2}}$$

$$PG = \frac{1}{3\sqrt{2}}$$

$h = \frac{\sin\theta}{3\sqrt{2}} + \frac{\cos\theta}{\sqrt{2}}$ & $k == \frac{\cos\theta}{3\sqrt{2}} + \frac{\sin\theta}{\sqrt{2}}$

$\sin\theta + 3\cos\theta = 3\sqrt{2}\,h$ &

$\Rightarrow \sin\theta + \cos\theta = 3\sqrt{2}\,k$

& $8\cos\theta = 3\sqrt{2}\,(3h - k)$

squaring & adding

$\Rightarrow 64 = 9.2\,[(h - 3k)^2 + (3h - k)^2]$

$\Rightarrow (3x - y)^2 + (x - 3y)^2 = \frac{32}{9}$

Sol.15 $BD = r_2$

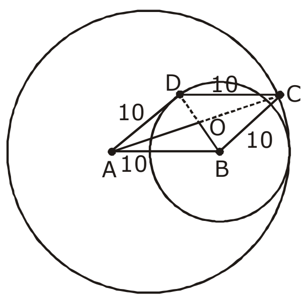
$AC = r_1$

$r_1 - r_2 = 10$

$\Rightarrow (r_1 - r_2)^2 - 2r_1r_2 = 100$

$\Rightarrow 2r_1r_2 = 400 - 100$

$\frac{r_1r_2}{2} = \frac{300}{4} = 75$ sq. units



In ΔOAB

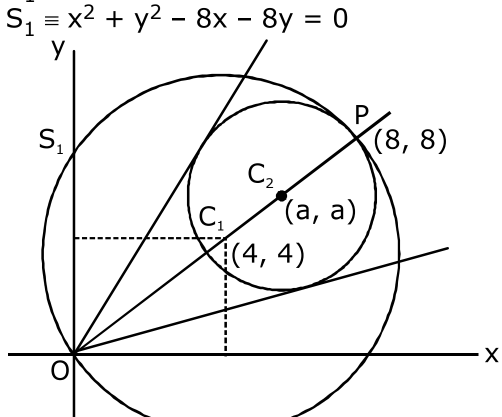
$\left(\frac{r_1}{2}\right)^2 + \left(\frac{r_2}{2}\right)^2 = 10^2$

$r_1^2 + r_2^2 = 400$

Sol.16 $OC_1 = 4\sqrt{2}$

$C_1 \equiv (4, 4)$

$S_1 \equiv x^2 + y^2 - 8x - 8y = 0$



Let $S_2 = x^2 + y^2 - 2ax - 2ay + c = 0$ ....(i)

{ $\because\ 7x^2 - 18xy + 7y^2 = 0$

angle bisector is $\frac{x^2 - y^2}{7 - 7} = \frac{xy}{-9}$

$\Rightarrow x = \pm y$

centres lie oin $x = y$

pair of tangent on S_2 from O

$SS_1 = T^2$

$c(x^2 + y^2 - 2ax - 2ay + c) = (-ax - ay + c)^2$

$\Rightarrow (c - a^2) - 2a^2xy + (c - a^2)\,y = 0$

by compaison

$\frac{c - a^2}{7} = \frac{-2a^2}{-18} \Rightarrow 9c - 9a^2 = 7a^2$

$\Rightarrow 9c = 16a^2$

S_1 & S_2 touches at (8, 8)

(8, 8) satisfy S_2

$128 - 32a + c = 0$

$\Rightarrow c = 32a - 128 \quad \therefore\ 9(32a - 128) = a^2$

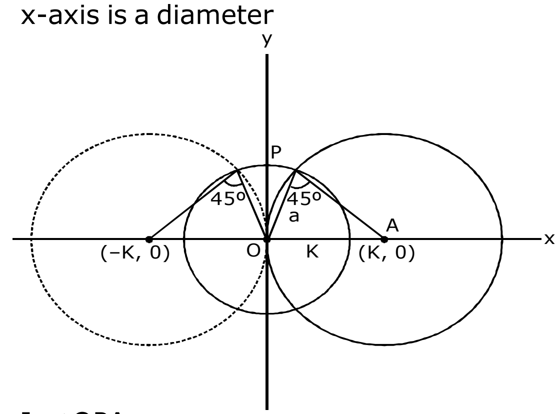
$\Rightarrow a^2 - 18a + 72 = 0 \Rightarrow a^2 - 18a + 72 = 0$

$\Rightarrow (a - 6)(a - 12) = 0$ $a = 6, 12$ S_1 & S_2 internally touches $\therefore$ $a = 6$ & $c = 64$

$\therefore$ circle $x^2 + y^2 - 12x - 12y + 64 = 0$

Sol.17 $x^2 + y^2 = a^2$

x-axis is a diameter



In ΔOPA

$\cos\theta = \cos 45º = \frac{a^2 + k^2 - k^2}{2ak}$

$\Rightarrow \sqrt{2}\,ak = a^2 \qquad \Rightarrow k = \frac{a}{\sqrt{2}}$

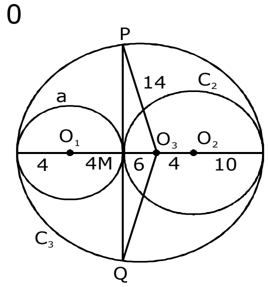
k can be (+) or (–)

$k = \pm\frac{a}{\sqrt{2}}$

$x^2 + y^2 \pm 2;\ \frac{a}{\sqrt{2}}\,x = 0$

$x^2 + y^2 \pm a\sqrt{2}\,x = 0$

Sol.18 $r_1 = 4,\ r_2 = 10$



$r_3 = \frac{2(r_1 + r_2)}{2}$

$r_3 = 14$

In ΔO_3MP

$O_3M = 6$

$PM = \sqrt{14^2 - 6^2} = \sqrt{160} = 4\sqrt{10}$

$PQ = 2PM \quad = \frac{8\sqrt{10}}{1} = \frac{m\sqrt{n}}{p}$

$\Rightarrow m + n + p = 8 + 10 + 1 = 19$